Symplectic Spinors, Holonomy and Maslov Index

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Abstract

In this note it is shown that the Maslov Index for pairs of Lagrangian Paths as introduced by Cappell, Lee and Miller ([1]) appears by parallel transporting elements of (a certain complex line-subbundle of) the symplectic spinorbundle over Euclidean space, when pulled back to an (embedded) Lagrangian submanifold L, along closed or non-closed paths therein. More precisely, the CLM-Index mod 4 determines the holonomy group of this line bundle w.r.t. the Levi-Civita-connection on L, hence its vanishing is equivalent to the existence of a trivializing parallel section. Moreover, it is shown that the CLM-Index determines parallel transport in that line-bundle along arbitrary endpoint-transverse paths, when compared to the parallel transport w.r.t. to the canonical flat connection of Euclidean space and also for certain elements of the dual spinor-bundle along closed or endpoint-transversal paths.

1 Introduction

The idea that (some kind of) Maslov Index is related to the double covering of the symplectic group, called the metaplectic group and to the notion of symplectic spinors has been implicit in the literature for quite a long time, mainly in the context of geometric quantization (see Guillemin [9], Kostant [12] and Crumeyrolle [3]). More recent work of Gosson ([7]), who gives an analytical definition of a maslov index (mapping to $\mathbb{Z}_4$) on the metaplectic group using its well-known Shale-Weil respresentation enlighted this area considerably. Using this and, to get in touch with some common definition of Maslov index, its link to the Maslov Index for pairs of Langrangian Paths as discussed by Cappell, Lee and Miller in their well known paper [1], the announced result is little more than 'piecing the edges together'. To give a brief outline of the argument, let (V, ω) be a fixed (finite dimensional) symplectic vectorspace and let $Lag(V)$ be the space of Lagrangian subspaces in V. To a continous and piecewise smooth path $f(t) = (L_1(t), L_2(t)), t \in [a, b]$ in $Lag(V) \times Lag(V)$ let the

$$\text{Maslov index} \quad \mu_{V,CLM}(f)$$

be the integer invariant associated to f following [1], from now on referred to as CLM-index. The CLM-index is characterized by a set of axioms which include homotopy invariance relative fixed endpoints which is the reason why there is an associated index $\mathcal{M}(x, y)$ for a pair (x, y) in the universal covering space $\pi : \widetilde{Lag}(V) \to Lag(V)$ of $Lag(V)$. In fact, choose a path $\tilde{\gamma} : [0, 1] \to \widetilde{Lag}(V)$ such that

$$\tilde{\gamma}(0) = x, \quad \tilde{\gamma}(1) = y.$$

If $\gamma = \pi\tilde{\gamma}$ then for any Lagrangian L_0 in V the integer

$$\mathcal{M}(L_0; x, y) = \mu_{V,CLM}([L_0], \gamma) \tag{1}$$

where $[L_0]$ is the constant path, is well-defined. One chooses $L_0 = \gamma(1)$, so from now on we refer to $\mathcal{M}(x, y) = \mathcal{M}([\gamma(1)]; x, y)$ as the Maslov-index on pairs of the universal covering space $\widetilde{Lag}(V)$.

Now, as we shall see below, the usual action of the symplectic group $Sp(V)$ of V on $Lag(V)$ is covered by an action of the universal covering group $\widetilde{Sp}(V)$ of $Sp(V)$ on $\widetilde{Lag}(V)$

$$\widetilde{Sp}(V) \times \widetilde{Lag}(V) \to \widetilde{Lag}(V).$$

For a fixed Lagrangian $L \in Lag(V)$ one now chooses an element $\tilde{L} \in \widetilde{Lag}(V)$ with $\pi(\tilde{L}) = L$ and observes that

$$m_L(\tilde{S}) = \mathcal{M}(\tilde{S}\tilde{L}, \tilde{L}) + n$$

where $\tilde{S} \in \widetilde{Sp}(V)$ does not depend on the choice of $\tilde{L}$ ($n = \frac{dim(V)}{2}$ is convention). Hence the last expression defines a $\mathbb{Z}$-valued mapping on $\widetilde{Sp}(V)$ associated to L. Finally, since $\widetilde{Sp}(V)$ covers the metaplectic group $\rho : Mp(V) \to Sp(V)$, say $\pi_2 : \widetilde{Sp}(V) \to Mp(V)$, one defines for $S \in Mp(V)$

$$m_L(S) = m_L(\tilde{S}) \mod 4$$

where $\pi_2(\tilde{S}) = S$ and shows that one gets a well defined mapping $m_L : Mp(V) \to \mathbb{Z}_4$. Now, specializing to (V, ω) as $(\mathbb{R}^{2n}, \omega_0)$, with ω_0 the symplectic standard structure on R^{2n}, Gosson ([7]) shows, that for $L_0 = \{0\} \times \mathbb{R}^n$ one recovers the index m_{L_0} on $Mp(2n, \mathbb{R})$ using analytic properties of the Shale-Weil-representation of the metaplectic group, this will be the key to our proof.
Given a Lagrangian embedding in $(\mathbb{R}^{2n}, \omega_0)$, that is a manifold L with $\dim(L) = n$ and an embedding $i : L \to \mathbb{R}^{2n}$ with $i^*\omega_0 = 0$, we will look at the pullback $i^*\mathcal{Q}_0$ of a certain complex one-dimensional subbbundle $\mathcal{Q}_0$ of $\mathcal{Q}$, the symplectic spinorbundle over $(\mathbb{R}^{2n}, \omega_0)$, to L and will consider the parallel transport $\mathcal{P}^{\nabla^g}$ in $i^*\mathcal{Q}_0$ induced by the Levi-Civita connection ∇^g of the Riemannian metric g on L which makes i isometric relative to the standard metric on $\mathbb{R}^{2n}$. It is known that

$$i^*\mathcal{Q}_0 \otimes i^*\mathcal{Q}_0 \simeq i^*\Lambda^{-1}, \tag{2}$$

where $i^*\Lambda$ denotes the canonical bundle on $\mathbb{R}^{2n}$, pulled back to L. We then have the following result.

Theorem 1.1. *For smooth closed paths $\gamma : [0,1] \to L$ based at $x \in L$ we have*

$$\mathcal{P}^{\nabla^g}_\gamma \varphi = e^{i\frac{\pi}{2}\mu_{CLM}([i_*T_xL],[i_*\gamma])}\varphi, \tag{3}$$

if $\varphi \in (\mathcal{Q}_0)_x$, where $[i_*T_xL]$ is the corresponding constant path and $[i_*\gamma]$ is the path $t \mapsto i_*(T_{\gamma(t)}L)$ in $Lag(\mathbb{R}^{2n})$. Consequently, for the holonomy group $\mathrm{Hol}^{\nabla^g}(i^*Q_0)$ we have $\mathrm{Hol}^{\nabla^g}(i^*Q_0) \subset \mathbb{Z}_4$.

Denote by $\mathrm{Par}^{\nabla^g}(i^*Q_0) \subset \Gamma(i^*\mathcal{Q}_0)$ the set of sections which are parallel w.r.t. ∇^g, Theorem 1.1 implies:

Corollary 1.2. *With the above notations we have $\dim_{\mathbb{C}}(\mathrm{Par}^{\nabla^g})(i^*Q_0) = 1$ if and only if $\mu_{CLM}([i_*T_xL],[i_*\gamma]) = 0 \mod 4$ for all $\gamma \in \pi_1(L)$.*

Note that from (2) it follows that the holonomy of $i^*\mathcal{Q}_0$ is determined by a Maslov-Index (namely the value of the mean-curvature form of L on $\gamma \in H_1(L, \mathbb{Z})$, see Oh [14]). However, our proof does not use (2) and instead derives the Theorem using the Maslov index $\hat{\mu}$ on $Mp(2n)$. Furthermore, the approach shows that μ_{CLM} determines parallel transport in $i^*\mathcal{Q}$ w.r.t. ∇^g along non-closed paths in L in an appropriate 'semi-classical limit'. To explain that, let ∇^0 denote the connection on $i^*\mathcal{Q}$ induced by the canonical flat connection on $\mathbb{R}^{2n}$, extended to the dual spinor bundle $i^*\mathcal{Q}'$. Assume $\delta_p(x) \in \mathcal{Q}'_x$ assigns for a given $p \in (P_L)_x$ in the $O(n)$-reduction of i^*P which is induced by L, P being the metaplectic structure of $(\mathbb{R}^{2n}, \omega_0)$ (c.f. Lemma 4.1), to any $\phi \in i^*\mathcal{Q}_x, \phi = [p, u]$ the value $\delta_p(\phi) = \delta(0)(u)$ (see (33)) and is extended to a ∇^0-parallel section $\delta_p \in \Gamma(i^*\mathcal{Q}')$, denote by $\delta_p(y) \in (i^*Q')_y$ its restriction to $y \in L$. Analogously, let $\mathbf{1}_p \in \Gamma(i^*\mathcal{Q}')$ be the ∇^0-parallel dual spinor field defined by $\mathbf{1}_p = [p, 1] \in i^*\mathcal{Q}'_x$. Then we have

Theorem 1.3. *Let $\gamma : [0,1] \to L$ denote a smooth path connecting $x, y \in L$ and assume that $i_*(T_xL) \cap i_*(T_yL) = 0$ in $\mathbb{R}^{2n}$. Then $\mathcal{P}_\gamma^{\nabla^g}\delta_p(x) \in (i^*\mathcal{Q}')_y$ and we have*

$$\mathcal{P}_\gamma^{\nabla^g}\delta_p(x) = c(y)e^{-i\frac{\pi}{2}\mu_{CLM}([i_*T_yL],[i_*\gamma])}\mathbf{1}_p(y), \tag{4}$$

for $0 < c(y) \in \mathbb{R}$ and $L(x) = i_*(T_xL)$ and $L(y) = i_*(T_xL)$, respectively. On the other hand, suppose that $\dim\, L(x) \cap L(y) = n$ *in $\mathbb{R}^{2n}$, then*

$$\mathcal{P}_\gamma^{\nabla^g}\delta_p(x) = e^{-i\frac{\pi}{2}\mu_{CLM}([i_*T_yL],[i_*\gamma])}\delta_p(y), \tag{5}$$

in $\mathcal{Q}'_y$.

Let now $\mathcal{Q}_l^J,\ l \in \mathbb{N}_0$ be the splitting of $\mathcal{Q}$ induced by the canonical complex structure J of $\mathbb{R}^{2n}$, i.e. $\mathcal{Q}_0 = \mathcal{Q}_0^J$ (see Section 3, Prop. 3.4). Then Theorem 1.3 immediately implies

Corollary 1.4. *Let $\gamma : [0,1] \to L$ be a smooth path with endpoints $x, y \in L$ s.t. $L(x) \cap L(y) = 0$ and let $\varphi \in \Gamma(i^*\mathcal{Q}_0)$ be ∇^0-parallel, then*

$$\mathcal{P}_\gamma^{\nabla^g}\varphi(x) = e^{i\frac{\pi}{2}\mu_{CLM}([i_*T_yL],[i_*\gamma])}\varphi(y). \tag{6}$$

On the other hand, if $\dim\, L(x) \cap L(y) = n$ *in $\mathbb{R}^{2n}$ and $\psi_l \in \Gamma(i^*\mathcal{Q}_l)$ is ∇^0-parallel then*

$$\delta_p(y)(\mathcal{P}_\gamma^{\nabla^g}\psi_l(x)) = e^{i\frac{\pi}{2}\mu_{CLM}([i_*T_yL],[i_*\gamma])}\delta_p(y)(\psi_l), \tag{7}$$

where $\delta_p \in \Gamma(\mathcal{Q}')$ is as defined above Theorem 1.3.

Note that (6) complements Theorem 1.1 to the case of endpoint-transversal paths in L. On the other hand, (7) means that μ_{CLM} determines the 'holonomy at zero' along closed paths in any of the subbundles $\mathcal{Q}_l$, that is, the holonomy multiplies the 'zero value' of any element of $\mathcal{Q}_l$ w.r.t to a given metaplectic frame by some element of $\mathbb{Z}_4 \subset U(1)$ which is determined by the Maslov index.
The paper is organized as follows: in Section 2, we will review in some more detail the above mentioned facts concerning the diverse integer invariants on Lagrangians paths and certain (cyclic) coverings of the symplectic group group. Section 3 contains a short discussion of the metaplectic representation with special emphasis on the properties of the so called 'quadratic Fourier transforms' and gives some necessary background on symplectic spinors. In section 4 finally we will arrive at the actual proof of Theorem 1.1 and 1.3.

2 Maslov indices for Lagrangian Paths and the Metaplectic Group

In this section, (V,ω) will be $(\mathbb{R}^{2n},\omega_0)$ and we will write $Lag(n)$, $\widetilde{Lag}(n)$ for the Lagrangian Grassmannian and its universal covering, $Sp(2n)$, $Mp(2n)$ and $\widetilde{Sp}(2n)$ for the symplectic group resp. its connected twofold and universal covering groups. To give some intuition to the definitions we will review some fundamental results about the Lagrangian Grassmannian, its universal covering and associated group actions, see for instance Souriau ([15]).
Since on one hand $U(n) = Sp(2n) \cap 0(2n)$ acts transitively on $Lag(n)$

$$U(n) \times Lag(n) \to Lag(n), \quad (s, L) \mapsto sL \tag{8}$$

with $O(n)$ the isotropy subgroup of $L_0 = 0 \times \mathbb{R}^\ltimes$ we have $Lag(n) \simeq U(n)/O(n)$. On the other hand, if $R_1, R_1 \in U(n)$ with $R_1L_0 = R_2L_0$ and the lower case letters r_1, r_2 denote the inverse images of R_1, R_2 under the isomorphism

$$i : U(n,\mathbb{C}) \subset M(n,\mathbb{C}) \to U(n) \subset M(2n,\mathbb{R}) \quad (A + iB) \mapsto \left(\begin{smallmatrix} A & -B \\ B & A \end{smallmatrix}\right)$$

where $A, B \in M(n, \mathbb{R})$, $A^T A + B^T B = I$ and $A^T B$ symmetric, then

$$R_1 L_0 = R_2 L_0 \;\Leftrightarrow\; r_1 (r_1)^T = r_2 (r_2)^T$$

where r^T is the transposed of r. Hence we get a homeomorphism

$$F : Lag(n) \to W(n, \mathbb{C}) = U(n, \mathbb{C}) \cap sym(n, \mathbb{C}), \quad L = RL_0 \mapsto rr^T$$

satisfying $F(RL) = rF(L)r^T$, concluding that we identified $Lag(n)$ with a subset of $U(n, \mathbb{C})$. Now the action (8) is covered by a unique transitive group action

$$\tilde{U}(n, \mathbb{C}) \times \widetilde{Lag}(n) \to \widetilde{Lag}(n) \tag{9}$$

where $\tilde{U}(n, \mathbb{C})$ is the universal covering group of $U(n, \mathbb{C})$ which can easily seen to be realized by defining

$$\tilde{U}(n, \mathbb{C}) = \{(r, \phi) : r \in U(n, \mathbb{C}),\ det(r) = e^{i\phi}\}$$

with the group composition $(r, \phi)(r', \phi') = (rr', \phi + \phi')$ and projection $\pi : (R, \phi) \mapsto R$ and using the topology induced by π. Now define $\widetilde{W}(n, \mathbb{C}) = \{(w, \phi) \in \tilde{U}(n, \mathbb{C}) : w \in W(n, \mathbb{C})\}$ with projection to $W(n, \mathbb{C})$ being the restriction of π and observe that $\widetilde{W}(n, \mathbb{C})$ is connected and simply connected since the group $\tilde{U}(n, \mathbb{C})$ acts transitively on $\widetilde{W}(n, \mathbb{C})$ with isotropy subgroup SO(n) of $(I, 0)$ by defining

$$(R, \phi)(w, \theta) = (rwr^T, \theta + 2\phi), \quad (r, \phi) \in \widetilde{U}(n, \mathbb{C}),\ (w, \theta) \in \widetilde{W}(n, \mathbb{C}). \tag{10}$$

So $\widetilde{W}(n, \mathbb{C}) \simeq \widetilde{Lag}(n)$ and the above action realizes (9) covering (8). The decktransformations of $\tilde{U}(n, \mathbb{C})$ are obviously of the form $I \times 2\pi\mathbb{Z}$, so $\pi_1(U(n, \mathbb{C})) = \pi_1(Sp(2n)) = I \times 2\pi\mathbb{Z}$. Identifying the group of decktransformations of $\widetilde{Lag}(n)$ with the subgroup $I \times \pi\mathbb{Z} \subset \widetilde{U}(n, \mathbb{C})$ by the action (10), we arrive at $\pi_1(Lag) = I \times \pi\mathbb{Z}$, if we denote $\beta = (I, \pi)$ and $\alpha = (I, 2\pi)$ the respective generators of $\pi_1(Lag(n))$ and $\pi_1(Sp(2n))$ we get

$$(\alpha\widetilde{U})(\tilde{L}) = \beta^2(\tilde{U}\tilde{L}) = \widetilde{U}(\beta^2\tilde{L}) \tag{11}$$

for $\tilde{U} \in \widetilde{U}(n, \mathbb{C})$, $\tilde{L} \in \widetilde{Lag}(n)$. So, understanding α resp. β as generators of the group of decktransformations of $\widetilde{Sp}(2n)$ and $\widetilde{Lag}(n)$ (using that $U(n, \mathbb{C}) \subset Sp(2n)$ is a maximal compact subgroup) we define for $q \in \mathbb{N}_+$

$$Sp_q(2n) = \widetilde{Sp}(2n)/\{\alpha^{qk} : k \in \mathbb{Z}\} \quad Lag_q(n) = \widetilde{Lag}(n)/\{\beta^{qk} : k \in \mathbb{Z}\}, \tag{12}$$

the (unique up to isomorphism) q-fold cyclic connected coverings $\rho_q : Sp_q(2n) \to Sp(2n)$ resp. $\varphi_q : Lag_q(n) \to Lag(n)$, that is $(\rho_q)_*(\pi_1(Sp_q(2n))) = q\mathbb{Z}$ resp. $(\varphi_q)_*(\pi_1(Lag_q(n))) = q\mathbb{Z}$ and one has the commuting diagram

$$\begin{array}{ccc} \widetilde{Sp}(2n) & \xrightarrow[\pi_q^{Sp}]{} & Sp_q(2n) \\ \Big\downarrow{\scriptstyle \pi^{Sp}} & & \Big\downarrow{\scriptstyle \rho_q} \\ Sp(2n) & \xrightarrow[id]{} & Sp(2n) \end{array} \tag{13}$$

where π_q is defined so that the diagram commutes. Note there is an analogous diagram in the case of $Lag(n)$ involving the mapping $\pi_k : \widetilde{Lag}(n) \to Lag_q(n)$ satisfying $\pi = \rho_q \circ \pi_q : \widetilde{Lag}(n)) \to Lag(n)$. As a consequence of (11), we infer that the action (9) projects for each $q > 0$ to an action

$$Sp_q(2n) \times Lag_{2q}(n) \to Lag_{2q}(n). \tag{14}$$

Now, in [8] resp. [7] one defines an index $\mu : \widetilde{Lag}(n) \times \widetilde{Lag}(n) \to \mathbb{Z}$ which is uniquely defined by the two conditions, where we write in the fowllowing $L = \pi(\tilde{L})$ for $\tilde{L} \in \widetilde{Lag}(n)$:

1. μ is locally constant on the set $\{(\tilde{L}_1, \tilde{L}_2) : L_1 \cap L_2 = 0\}$
2. for $(\tilde{L}_1, \tilde{L}_2, \tilde{L}_3) \in \widetilde{Lag}^3(n)$ we have

$$\mu(\tilde{L}_1, \tilde{L}_2) - \mu(\tilde{L}_1, \tilde{L}_3) + \mu(\tilde{L}_2, \tilde{L}_3) = \tau(L_1, L_2.L_3),$$

where τ is the signature of the quadratic form on $L_1 \oplus L_2 \oplus L_3$ defined by

$$(z_1, z_2, z_3) \mapsto \omega(z_1, z_2) \oplus \omega(z_2, z_3) \oplus \omega(z_1, z_3).$$

As is shown in ([8], Proposition 3.16 resp. Corollary 3.22), if $(\tilde{S}, \tilde{L}_1, \tilde{L}_2) \in Sp_q(2n) \times Lag_{2q}(n)^2$, then

$$\mu(\tilde{S}\tilde{L}_1, \tilde{S}\tilde{L}_2) = \mu(\tilde{L}_1, \tilde{L}_2), \tag{15}$$

furthermore if $\beta = (I, \pi)$ generates $\pi_1(Lag(n))$ as above, then

$$\mu(\beta^r \tilde{L}_1, \beta^{r'} \tilde{L}_2) = \mu(\tilde{L}_1, \tilde{L}_2) + 2(r - r'), \tag{16}$$

if $r, r' \in \mathbb{Z}$. For $\tilde{L}_1, \tilde{L}_2$, let $\mathcal{M}(\tilde{L}_1, \tilde{L}_2) \in \mathbb{Z}$ be as defined below (1) and define

$$\hat{\mu}(\tilde{L}_1, \tilde{L}_2) = 2\mathcal{M}(\tilde{L}_1, \tilde{L}_2) + (n - \dim(L_1 \cap L_2)). \tag{17}$$

Then using the defining conditions for μ, it is proven in ([1], Prop. 9.1) that

Lemma 2.1. *For all $\tilde{L}_1, \tilde{L}_2 \in \widetilde{Lag}(n)$ the index $\mu(\tilde{L}_1, \tilde{L}_2)$ coincides with $\hat{\mu}(\tilde{L}_1, \tilde{L}_2)$.*

The two properties (15) and (16) of μ imply the following Proposition resp. Definition of a Maslov index on $\widetilde{Sp}(2n)$ resp. $Sp_q(2n)$ relative to a fixed Lagrangian $L \in Lag(n)$, which was the aim of this section:

Lemma 2.2. *Let $L \in Lag(n)$, then the mapping $\mu : \widetilde{Sp}(2n) \to \mathbb{Z}$ given by*

$$\mu_L(\tilde{S}) = \mu(\tilde{S}\tilde{L}, \tilde{L})$$

is well-defined, that is, independent of the choice of $\tilde{L}$ lifting L. Furthermore, for any $q \in \mathbb{N}_+$, $\mu(\cdot)$ mod $4q$ factorizes to a well-defined mapping $\mu_q : Sp_q(2n) \to \mathbb{Z}_{4q}$, that is for $S_q \in Sp_q(2n)$ the expression

$$\mu_{L,q}(S_q) = \mu(\tilde{S}\tilde{L}, \tilde{L}) \ \mathit{mod}\ 4q$$

so that $\pi_q^{Sp}(\tilde{S}) = S_q$ does not depend on the choice of $\tilde{S} \in \widetilde{Sp}(2n)$.

Proof. The proof is given in Gosson's book [8] and follows directly by invoking the properties (15) and (16) of μ on $\widetilde{Lag}(n)^2$ and by noting that these together with (11) imply for $r \in \mathbb{Z}$ and $\tilde{S} \in \widetilde{Sp}(2n)$ and with $\alpha = (I, 2\pi)$ generating $\pi_1(Sp(2n))$ as above

$$\mu_L(\alpha^r \tilde{S}) = \mu_L(\alpha^r \tilde{S}) + 4r.$$

□

Combining the preceeding Lema and Lemma 2.1, we arrive at

Corollary 2.3. *Let $S : [0,1] \to Sp(2n)$ be piecewise smooth, $S(0) = Id$, let $L \in Lag(n)$ be arbitrary and let $\hat{S} : [0,1] \to Mp(2n) = Sp_2(2n)$ be the unique lift of S that begins at $Id \in Mp(2n)$, that is $\rho(\hat{S}(t)) := \rho_2(\hat{S}(t)) = S(t)$, $t \in [0,1]$ and $\hat{S}(0) = Id$. Denote $L(t) = S(t)L \in Lag(n)$, $t \in [0,1]$. Then one has*

$$\mu_{L,2}(\hat{S}(1)) = 2\mu_{CLM}([L(1)], L(t)) + (n - \dim(L(0) \cap L(1)) \bmod 8, \tag{18}$$

where μ_{CLM} is the Maslov index on pairs of Lagrangian paths introduced in (1) in $(\mathbb{R}^{2n}, \omega_0)$.

Proof. Let $\tilde{L} \in \widetilde{Lag}(n)$ be any element covering $L(0)$, so $\pi(\tilde{L}) = L(0)$. Let $\tilde{S}$ be the element in $\widetilde{Sp}(2n)$ defined by the homotopy class of $S : [0,1] \to Sp(2n),\ S(0) = Id$, then by (13) we have

$$\pi_2^{Sp}(\tilde{S}) = \hat{S}(1). \tag{19}$$

On the other hand, denoting the lift of $S(t)$ to $\widetilde{Sp}(2n)$ by $\tilde{S} : [0,1] \to \widetilde{Sp}(2n)$, we have since $\tilde{S}(1) = \tilde{S}$ that $\tilde{S}(t)\tilde{L} \in \widetilde{Lag}(n)$ connects $\tilde{L}$ in $\widetilde{Lag}(n)$ to $\tilde{S}\tilde{L}$ and projects to $S(t)L$, that is $\pi(\tilde{S}(t)\tilde{L}) = S(t)L \in Sp(2n)$. Using (19) and the latter observations together with (17), Lemma 2.1, Lemma 2.2 and the relation between μ_{CLM} and $\mathcal{M}(\cdot,\cdot)$ expressed in (1) we arrive at (18). $\square$

3 The Metaplectic Representation and the Symplectic Spinor bundle

As we saw in the last section, $\pi_1(Sp(2n)) = \mathbb{Z}$, this implies since there is only one conjugation class of subgroups of index 2 in $\mathbb{Z}$, that there is up to isomorphism exactly one connected two-fold covering $\rho : Mp(2n) \to Sp(2n)$, fitting into the sequence

$$1 \quad \to \quad \mathbb{Z}_2 \quad \to \quad Mp(2n) \quad \xrightarrow{\rho} \quad Sp(2n) \quad \to \quad 1.$$

Consequently, $Mp(2n) \simeq Sp_2(2n)$ (we will prefer the notation $Mp(2n)$ in the context of its 'metaplectic' representation, described in what follows) carries a unitary, faithful representation $\kappa : Mp(2n) \to \mathcal{U}(L^2(\mathbb{R}^n))$ which can be constructed by lifting the projective representation of $Sp(2n)$ induced by intertwining the Schroedinger representation of the Heisenberg group to $Mp(2n)$, for more details of this, see Wallach ([16]). This representation κ has the following explicit construction on the elements of three generating subgroups of $Mp(2n,\mathbb{R})$, as follows:

1. Let $g(A) = (det(A)^{\frac{1}{2}}, \begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix})$ where $A \in GL(n,\mathbb{R})$. To fix a root of $det(A)$ defines $g(A)$ as an element in $Mp(2n)$ and we have

$$(\kappa(g(A))f)(x) = det(A)^{\frac{1}{2}} f(A^t x),\ f \in L^2(\mathbb{R}^n). \tag{20}$$

2. Let $B \in M(n,\mathbb{R})$ mit $B^t = B$, so that $t(B) = \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} \in Sp(2n)$ then the set of these matrices is simply-connected. So $t(B)$ can be considered an element of $Mp(2n)$, with $t(0)$ being the identity in $Mp(2n)$. The one has

$$(\kappa(t(B))f)(x) = e^{-\frac{i}{2}\langle Bx,x\rangle} f(x). \tag{21}$$

3. Fixing the root $i^{\frac{1}{2}}$ we have $\sigma = (i^{\frac{1}{2}}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}) \in Mp(2n)$. Then

$$(\kappa(\sigma)f)(x) = (\frac{i}{2\pi})^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\langle x,y\rangle} f(y)dy, \tag{22}$$

so $\kappa(\sigma) = i^{\frac{n}{2}} F^{-1}$, where F is the usual Fourier transform.

Inspecting these formulas it is obvious that the metaplectic group $Mp(2n)$ acts bijectively on the Schwartz space $\mathcal{S}(\mathbb{R}^n)$, so its closure extends to $\mathcal{U}(L^2(\mathbb{R}^n))$. We then have the following Theorem due to Wallach [16] (p. 193, Theorem 4.53) which gives a description of κ on a certain subset of $Mp(2n)$ in terms of oscillatory integrals:

Theorem 3.1. *let $\mathcal{A} = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in Sp(2n)$ s.t. $\det B \neq 0$, then*

$$\hat{S}_{W,m} := \kappa((\det{}^{-1/2} B, \mathcal{A})) = (\frac{1}{2\pi i})^{n/2} |\det(B)|^{-\frac{1}{2}} i^m \int_{\mathbb{R}^n} e^{2\pi i W(x,y)} f(x') dx', \tag{23}$$

where $W(x, x')$ is the generating function associated to $\mathcal{A}$, that is $(x, p) = \mathcal{A}(x', p')$ if and only if $p = \partial_x W(x, x')$, $p' = -\partial_{x'} W(x, x')$.

Note that here we fixed the root $i^{n/2} = (e^{i\pi/4})^n$ while the choice of the root $\det^{-1/2}(B)$ fixes the element in $Mp(2n)$ covering $\mathcal{A}$. Denote now $\hat{J} := \kappa(\sigma^{-1}) = i^{-\frac{n}{2}} F$, where we fix again $i^{n/2} = (e^{i\pi/4})^n$. Furthermore, write for A as in (20) $\kappa(g(A), m) = |\det(A)|^{1/2} i^m f(A^t x)$, where $m \in \mathbb{Z}$ and $|\det(A)|^{1/2}$ (as already in (23)) denotes the positive root of $|\det(A)|$. Then for $P, Q \in \mathrm{M}(n, \mathbb{R})$, s.t. $P = P^t$, $Q = Q^t$ and $L \in \mathrm{GL}(n, \mathbb{R})$ we define the quadratic form

$$W(x, x') = \frac{1}{2}\langle Px, x\rangle - \langle Lx, x'\rangle + \frac{1}{2}\langle Qx', x'\rangle, \tag{24}$$

where $< \cdot, \cdot >$ denotes the standard scalar product on $\mathbb{R}^n$. We will use the notation $W = (P, L, Q)$ to refer to a quadratic form of type (24) in the following. Then, by a result of Gosson ([7], Prop. 7.2), the 'quadratic Fourier transform' $\hat{S}_{W,m}$ can be decomposed as

$$\hat{S}_{W,m} = \kappa((\det{}^{1/2}(L), S_W) = \kappa(t(P))\kappa(g(L), m)\hat{J}\kappa(t(Q)), \text{ where } S_W := \left(\begin{smallmatrix} L^{-1}Q & L^{-1} \\ PL^{-1}Q - L^T & L^{-1}P \end{smallmatrix}\right), \tag{25}$$

where here, $|\det(L)|^{1/2} i^m = \det^{1/2}(L)$. The next Theorem identifies the Maslov index $\mu_{L_0,2}$ on $Sp_2(2n)$, where $L_0 = \{0\} \times \mathbb{R}^n$, as introduced in Lemma 2.2, with an index defined on the group generated by the set $\hat{S}_{W,m}$, for W as in (24), which turns out to be $Mp(2n)$.

Theorem 3.2. *The image $\kappa(Mp(2n)) \subset \mathcal{U}(L^2(\mathbb{R}^n))$ is generated by the set $\hat{S}_{W,m}$, W being of the form (24). Any element $\hat{S} \in Mp(2n)$ can be (non-uniquely) written as*

$$\hat{S} = \hat{S}_{W,m}\hat{S}_{W',m'}, \tag{26}$$

where W, W are of the form (24). Then setting $\hat{\mu}(\hat{S}_{W,m}) = 2m - n \bmod 8$ for any 'quadratic Fourier transform' $\hat{S}_{W,m}$ as defined in (23), the integer

$$\hat{\mu}(\hat{S}) := \hat{\mu}(\hat{S}_{W,m}) + \hat{\mu}(\hat{S}_{W',m'}) + \widehat{\mathrm{sign}}(P' + Q) \tag{27}$$

where $\hat{(\cdot)}$ denotes the image in $\mathbb{Z}_8$ and sign *the signature of a quadratic form, is well-defined and independent of the choice of $(W, m), (W', m')$. Furthermore, assuming that $\hat{S} \in \kappa(Mp(2n))$ maps to $S_2 \in Sp_2(2n)$ w.r.t. the identfication $\kappa(Mp(2n)) \simeq Sp_2(2n)$, we have*

$$\hat{\mu}(\hat{S}) = \mu_{L_0,2}(S_2), \tag{28}$$

using the index $\mu_{L_0,2} : Sp_2(2n) \to \mathbb{Z}_8$ introduced in Lemma 2.2.

Proof. That $\kappa(Mp(2n))$ is generated by the 'quadratic Fourier transforms' $\hat{S}_{W,m}$ follows immediately from the decomposition (25) and the formulas given for κ in (20) to (22). All other assertions, namely (26), (27) and (28) are proven by Gosson in [7] (Prop. 7.2, Theorem 7.22 and Corollary 7.30, respectively). □

Let now (M, ω) be a symplectic manifold of dimension $2n$. For $p \in M$ we denote by R_p the set of symplectic bases in T_pM, that is the $2n$-tuples $e_1, \dots, e_n, f_1, \dots, f_n$ so that

$$\omega_x(e_j, e_k) = \omega_x(f_j, f_k) = 0, \ \omega_x(e_j, f_k) = \delta_{jk} \quad \text{for } j, k = 1, \dots, 2n.$$

The symplectic group $Sp(2n)$ acts simply transitively on $R_p,\ p \in M$ and we denote by $\pi_R : R := \bigcup_{p\in m} R_p \to M$ the symplectic frame bundle, by the Darboux Theorem R it is a locally trivial $Sp(2n)$-principal fibre bundle on M. As it is well-known (see [5]), the ω-compatible almost complex structures J are in bijective correspondence with the set of $U(n)$-reductions of R. Given such a J, we call local sections of the associated $U(n)$-reduction R^J of the form $(e_1,\ldots,e_n,f_1,\ldots,f_n)$ unitary frames, they are characterized by

$$g(e_j,e_k)=\delta_{jk} \quad g(e_j,f_k)=0, \quad Je_j=f_j,$$

where $j,k=1,\ldots,n$ and $g(\cdot,\cdot)=\omega(\cdot,J\cdot)$. Now a metaplectic structure of (M,ω) is a ρ-equivariant $Mp(2n)$-reduction of R, that is:

Definition 3.3. *A pair (P,f), where $\pi_P : P \to M$ is a $Mp(2n,\mathbb{R})$-principal bundle on M and f a mapping $f : P \to R$, is called metaplectic structure of (M,ω), if the following diagram commutes:*

$$\begin{tikzcd} P \times Mp(2n,\mathbb{R}) \arrow[r] \arrow[d, "f\times\rho"] & P \arrow[d, "f"] \\ R \times Sp(2n,\mathbb{R}) \arrow[r] & R \end{tikzcd} \tag{29}$$

where the horizontal arrows denote the respective group actions.

It follows that $f : P \to R$ is a two-fold connected covering, furthermore it is known ([10], [12]) that (M,ω) admits metaplectic structure if and only if $c_1(M) = 0$ *mod* 2. In that case, the isomorphy classes of metaplectic structures are classified by $H^1(M,\mathbb{Z}_2)$. Fixing a metaplectic structure P over M, κ induces a continous left-action on $L^2(\mathbb{R}^n)$, since κ is continous w.r.t. to the strong topology on $\mathcal{U}(L^2(\mathbb{R}^n))$. Combining this with the right-action of $Mp(2n)$ on P, we get a continous right-action on $P \times L^2(\mathbb{R}^n)$ by setting

$$\begin{aligned} (P\times L^2(\mathbb{R}^n)) \times Mp(2n) &\to P\times L^2(\mathbb{R}^n) \\ ((p,f),g) &\mapsto (pg,\kappa(g^{-1})f). \end{aligned}$$

and the symplectic spinor bundle $\mathcal{Q}$ is defined to be its orbit space:

$$\mathcal{Q} = P \times_\kappa L^2(\mathbb{R}^n) := (P\times L^2(\mathbb{R}^n))/Mp(2n)$$

w.r.t. this group action, so $\mathcal{Q}$ is the κ-asscoiated vector bundle of P, we will refer to its elements in the following by $[p,u]$, $p\in P$, $u \in L^2(\mathbb{R}^n)$. Note that if π_P is the projection $\pi : P \mapsto M$ in P, then $\mathcal{Q}$ is a locally trivial fibration $\tilde{\pi} : \mathcal{Q} \to M$ with fibre $L^2(\mathbb{R}^n)$ by setting $\tilde{\pi}([p,u]) \mapsto x$ if $\pi_P(p) = x$. Note that continous sections ϕ in $\mathcal{Q}$ correspond to $Mp(2n)$-equivariant mappings $\hat{\phi} : P \to L^2(\mathbb{R}^n)$, that is $\hat{\phi}(pq) = \kappa(q^{-1})\hat{\phi}(p)$ for $p \in P$, which is why we define *smooth* sections $\Gamma(\mathcal{Q})$ in $\mathcal{Q}$ as the continous sections whose corresponding mapping $\hat{\phi}$ is smooth as a map $\hat{\phi} : P \to L^2(\mathbb{R}^n)$, it then follows ([10]) that $\hat{\phi}(p) \in \mathcal{S}(\mathbb{R}^n)$ for all $p \in P$, so smooth sections in $\mathcal{Q}$ are in fact sections of the subbundle

$$\mathcal{S} = P \times_\kappa \mathcal{S}(\mathbb{R}^n).$$

Given a $U(n)$-reduction R^J of R w.r.t. a compatible almost complex structure J on M and a fixed metaplectic structure P, we get a $\hat{U}(n) := \rho^{-1}(U(n))$-reduction P^J of P, by setting $P^J := f^{-1}(R^J)$, where f is as in Definition 3.3, so we get by restricting κ to $\tilde{\kappa}$ on $\hat{U}(n)$

$$\mathcal{Q} = \mathcal{Q}^J := P^J \times_{\tilde{\kappa}} L^2(\mathbb{R}^n). \tag{30}$$

At this point, the Hamilton operator H_0 of the harmonic oscillator on $L^2(\mathbb{R}^n)$ gives rise to an endomorphism of $\mathcal{S}$ and a splitting of $\mathcal{Q}$ into finite-rank subbundles as follows. Let $H_0 : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ be the Hamilton operator of the n-dimensional harmonic oscillator as given by

$$(H_0u)(x) = -\frac{1}{2}\sum_{j=1}^n (x_j^2 u - \frac{\partial^2 u}{\partial x_j^2}),\ u \in \mathcal{S}(\mathbb{R}^n).$$

Proposition 3.4. *The bundle endomorphism $\mathcal{H}^J : \mathcal{S} \to \mathcal{S}$ declared by $\mathcal{H}^J([p,u]) = [p, H_0 u]$, $p \in P, u \in \mathcal{S}(\mathbb{R}^n)$ is well-defined. Let $\mathcal{M}_l$ denote the eigenspace of H_0 with eigenvalue $-(l+\frac{n}{2})$. Then the spaces $\mathcal{M}_l$, $l \in \mathbb{N}_0$ form an orthogonal decomposition of $L^2(\mathbb{R}^n)$ which is $\tilde{\kappa}$-invariant. So $\mathcal{Q}$ decomposes into the direct sum of finite rank-subbundles*

$$\mathcal{Q}_l^J = P^J \times_{\tilde{\kappa}} \mathcal{M}_l, \quad \text{s.t. } \mathrm{rank}_{\mathbb{C}} \mathcal{Q}_k^J = \binom{n+k-1}{k}$$

where we defined $\mathcal{Q}_l^J = \{q \in \mathcal{S} : \mathcal{H}^J(q) = -(l+\frac{n}{2})q\}$.

Proof. It is well-known (see [16], [10]) that H_0 can be identified with the element $j \in \mathfrak{mp}(2n)$, where $\mathfrak{mp}(2n)$ denotes the Lie-Algebra of $Mp(2n)$, that satisfies $\rho_*(j) = -J \in \mathfrak{sp}(2n)$, where J denotes the standard complex structure on $\mathbb{R}^{2n}$. Then one sees that J commutes with all elements of the Lie-Algebra of $U(n)$, as given by

$$\mathfrak{u}(n) = \{X \in \mathfrak{gl}(2n, \mathbb{R}) : XJ = JX,\ X^t + X = 0\}. \tag{31}$$

Consequently, H_0 factors to a bundle endomorphism $\mathcal{H}^J$ and the other assertions follow from known results on the eigenspaces of H_0 on $L^2(\mathbb{R}^n)$ (see [16]). $\square$

To prove Theorem 1.3, we will have to define the dual spinor bundle Q' of Q. To do this, note that if we topologize the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ by the countable family of semi-norms

$$p_{\alpha,m}(f) = \sup_{x\in\mathbb{R}^n}(1+|x|^m)|(D^\alpha f)(x)|,\ f \in \mathcal{S}(\mathbb{R}^n),$$

then $\kappa : Mp(2n) \to \mathcal{U}(\mathcal{S}(\mathbb{R}^n))$ still acts continously, which follows by the decomposition (20)-(22) and the fact that multiplication by monomials and Fourier transform act continously w.r.t. τ, which is a standard result (see [17]). The topology of $(\mathcal{S}(\mathbb{R}^n), \tau)$ is induced by a translation-invariant complete metric, hence manifests the structure of a Frechet-space. Then, denoting the dual space of $(\mathcal{S}(\mathbb{R}^n), \tau)$ as $\mathcal{S}'(\mathbb{R}^n)$, we can consider for any pair $T \in \mathcal{S}'(\mathbb{R}^n), g \in Mp(2n)$ the continous linear functional $\kappa(g)(T) \in \mathcal{S}'(\mathbb{R}^n)$ defined by

$$(\hat{\kappa}(g)(T))(f) = T(\kappa(g)^* f),\ f \in \mathcal{S}(\mathbb{R}^n), \tag{32}$$

that is, we have an action $\hat{\kappa} : Mp(2n) \times \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ which extends $\kappa : Mp(2n) \to \mathcal{U}(\mathbb{R}^n)$ and is continous relative to the weak-$*$-topology on $\mathcal{S}'(\mathbb{R}^n)$. Note that since the inclusion $i_1 : \mathcal{S}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$ is continous, we have the continous triple of embeddings $\mathcal{S}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$, where $L^2(\mathbb{R}^n)$ carries the norm topology and the inclusion $i_2 : L^2(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$ is given by $i_2(f)(u) = (f, \overline{u})_{L^2(\mathbb{R}^n)}$ where the latter denotes the usual L^2-inner product on $\mathbb{R}^n$. We thus define in analogy to (30)

$$\mathcal{Q}' = P^J \times_{\hat{\kappa}} \mathcal{S}'(\mathbb{R}^n),$$

where here, $\hat{\kappa} : U(n) \to \mathcal{S}'(\mathbb{R}^n)$ means restriction of $\hat{\kappa}$ to $U(n)$ (using the same symbol). Now any fixed section $\varphi \in \Gamma(\mathcal{Q}')$ may be evaluated on any $\psi \in \Gamma(\mathcal{Q})$ by writing $\varphi = [\overline{s}, T], \psi = [\overline{s}, u]$ w.r.t. a local section $s : U \subset M \to P^J$ and smooth mappings $T : U \to \mathcal{S}'(\mathbb{R}^n)$, $u : U \to \mathcal{S}(\mathbb{R}^n)$ and setting

$$\varphi(\psi)|U = T(u)(x),\ x \in U \subset M.$$

It is clear that this extends to a mapping $\varphi : \Gamma(\mathcal{Q}) \to C^\infty(M)$. Furthermore, for any $p \in P^J, x \in M$ s.t. $\pi_{P^J}(p) = x$, we can define $\delta_p \in \mathcal{Q}'_x$ which assigns to any $\psi = [p, u] \in \mathcal{Q}_x$ the value

$$\delta_p(\psi) := \delta(0)(u), \tag{33}$$

where $\delta(0) \in \mathcal{S}'(\mathbb{R}^n)$ is the linear functional $\delta_0(u) = u(0)$, $u \in \mathcal{S}(\mathbb{R}^n)$. Note that δ_p depends on p and, unless P^J has a global section, there is not necessarily a smooth extension of δ_p to an element

of $\Gamma(\mathcal{Q}')$ so that over any point $x \in M$ (33) holds for some $p \in P^J_x$. Nevertheless, given some connection $\overline{Z} : P^J \to \mathfrak{U}(n)$ the associated parallel transport $\mathcal{P}^{\overline{Z}}_\gamma(t) : P^J_{\gamma(0)} \to P^J_{\gamma(t)},\ t \in [0,1]$ along $\gamma : [0,1] \to M$ enables to extend δ_p along γ to a section $\delta_{\gamma,p} \in \Gamma(\gamma^*(\mathcal{Q}'))$ by setting

$$\delta_{\gamma,p}(t) = [\mathcal{P}^{\overline{Z}}_\gamma(t)(p), \delta(0)] \in \gamma^*(\mathcal{Q}')_{\gamma(t)},$$

we will return to that in the next section.

4 Proof of the Theorems

In the following, let $(M,\omega) = (\mathbb{R}^{2n}, \omega_0)$, using the notation from Section 2 and let $i : L \hookrightarrow \mathbb{R}^{2n}$, be an embedded Lagrangian submanifold, that is $i^*\omega = 0$. Denote J the standard complex structure on $\mathbb{R}^{2n}$ and Q^J the symplectic spinor bundle associated to the $\hat{U}(n)$-reduction $f^J : P^J \to R^J$ of the trivial metaplectic structure $f : P \to R$ on $\mathbb{R}^{2n}$ (note that since $c_1(\mathbb{R}^{2n}, \omega) = 0$, there is only this structure up to isomorphism). We first note that the bundles $\pi^L_R : i^*R^J \to L$ resp. $\pi^L_P : i^*P^J \to L$ allow a further reduction to $O(n)$ resp. $\hat{O}(n) = \rho^{-1}(O(n))$ induced by the inclusion

$$i : O(n) \hookrightarrow U(n) = Sp(2n) \cap O(2n), \quad A \mapsto \left(\begin{smallmatrix} A & 0 \\ 0 & A \end{smallmatrix}\right), \tag{34}$$

where $A \in M(n,\mathbb{R}),\ A^tA = I$. Denote $g(\cdot,\cdot) = \omega(\cdot, J\cdot)$ the metric induced on L by (J,ω), which is simply the restriction of the standard metric to L.

Lemma 4.1. *There is an $O(n)$-reduction $(\hat{R}_L, \pi^L_R, L, O(n))$ of the principal bundle $(i^*R^J, \pi_R|L, L, U(n))$ which is induced by the inclusion (34). This reduction gives rise to an $\hat{O}(n) = \rho^{-1}(O(n))$-reduction $(\hat{P}_L, \pi^L_P, L, \hat{O}(n))$ of $(i^*P^J, \pi_P|L, L, \hat{U}(n))$ on L by setting $\hat{P}_L = f^{-1}(\hat{R}_L)$ so that the diagram*

$$\begin{array}{ccc} \hat{P}_L \times \hat{O}(n) & \longrightarrow & \hat{P}_L \\ \Big\downarrow{\scriptstyle \hat{f}\times\hat{\rho}} & & \Big\downarrow{\scriptstyle \hat{f}} \\ \hat{R}_L \times O(n) & \longrightarrow & \hat{R}_L \end{array} \tag{35}$$

commutes. Here, $\hat{f}$ and ρ denote appropriate restrictions of $f : P \to R$ and $\rho : Mp(2n) \to Sp(n)$ as defined above.

Proof. To show that the $U(n)$-bundle $\hat{R} := i^*R^J$ over L allows the asserted $O(n)$-reduction, we have to show that the bundle $\hat{R} \times_{U(n)} U(n)/O(n)$ allows a global section over L. But this is determined by setting locally for $v \in U \subset L$

$$\phi(x) = [s(x), 1],\ x \in L,$$

where $s(x) \in \hat{R}, \pi_R(s(x)) = x$ and $s(x) = (e_1(x), \ldots, e_n(x), Je_1, \ldots, e_n(x))$, where $(e_1, \ldots, e_n)$ is some local orthonormal basis on L and 1 denotes the identity in $U(n)/O(n)$. Is is clear that ϕ declares a well-defined globally non-vanishing section of $\hat{R} \times_{U(n)} U(n)/O(n)$. $\square$

Let now $Z^L : TR_L \to \mathfrak{o}(n,\mathbb{R})$ be the connection on R_L, where R_L is the $O(n)$-bundle of orthonormal frames on (L,g), which corresponds to the Levi-Civita covariant derivative ∇^g on (L,g). Then it is clear that if $j : R_L \to \hat{R}_L$ is the fibre bundle isomorphism given by setting for any $x \in L$ $j(e_1, \ldots, e_n) = (e_1, \ldots, e_n, Je_1, \ldots, Je_n)$, where $(e_1, \ldots, e_n) \in (R_L)_x,\ x \in L$ is an orthonormal basis in T_xL, that

$$Z : T\hat{R}_L \to \mathfrak{o}(n), \quad Z := i_* \circ Z^L \circ (j_*)^{-1}, \tag{36}$$

defines a well-defined connection on $\hat{R}_L$. Furthermore, Z lifts to a connection 1-form $\overline{Z} : T\hat{P}_L \to \hat{\mathfrak{o}}(n)$, so that the following diagram commutes, here we set $\hat{\mathfrak{o}}(n) = \rho_*^{-1}(\mathfrak{o}(n))$:

$$\begin{array}{ccc} T\hat{P}_L & \xrightarrow{\overline{Z}} & \hat{\mathfrak{o}}(n) \\ \downarrow{\scriptstyle \hat{f}_*} & & \downarrow{\scriptstyle \hat{\rho}_*} \\ T\hat{R}_L & \xrightarrow{Z} & \mathfrak{o}(n) \end{array}$$

Since ρ_* is an isomorphism, we can actually define $\overline{Z}$ as $\overline{Z} = \rho_*^{-1} \circ Z \circ f_*$ on $T\hat{P}_L$. Note that using the above, $i^*\mathcal{Q}_l^J,\ l \in \mathbb{N}_0$ can be written as

$$i^*\mathcal{Q}_l^J = \hat{P}_L \times_{\hat{\kappa}} \mathcal{M}_l,$$

where $\hat{\kappa} = \kappa|\hat{O}(n)$. For $s : U \subset L \to \hat{R}_L$ a local section in $\hat{R}_L$, let $\overline{s} : U \subset L \to P_L$ be a lift to $\hat{P}_L$. Then if $X \in \Gamma(TL)$, $\overline{Z}$ induces a covariant derivative in $\Gamma(i^*\mathcal{Q}^J)$ by setting for a local section $\varphi = [\overline{s}, u]$, where $u : U \to L^2(\mathbb{R}^n)$

$$\nabla_X\varphi = [\overline{s}, du(X) + \hat{\kappa}_*(\overline{Z} \circ \overline{s}_*(X))u].$$

On the other hand, given a path $\gamma : [0,1] \to L$, $\gamma(0) \in U$, the horizontal lift γ_p of γ w.r.t. $\overline{Z}$ and a given $\gamma_p(0) = p \in (P_L)_{\gamma(0)}$ defines a map $\mathcal{P}_\gamma^{\overline{Z}}(t) : (\hat{P}_L)_{\gamma(0)} \to (\hat{P}_L)_{\gamma(t)}$ by setting $\mathcal{P}_\gamma^{\overline{Z}}(t)(p) = \gamma_p(t)$ which in turn defines the notion of parallel transport $\mathcal{P}_\gamma^{\nabla}(t) : (i^*\mathcal{Q}_0^J)_{\gamma(0)} \to (i^*\mathcal{Q}_0^J)_{\gamma(t)}$ by setting

$$\mathcal{P}_\gamma^{\nabla}(t)[p,u] = [\mathcal{P}_\gamma^{\overline{Z}}(t)(p), u],\ u \in \mathcal{M}_0,\ t \in [0,1]$$

and if $\varphi \in \Gamma(i^*\mathcal{Q}_0^J)$ and $\gamma'(0) = X$, then $\nabla_X\varphi = \frac{d}{dt}(\mathcal{P}_\gamma^{\nabla}(t)(\varphi(\gamma(t))))|_{t=0}$. Note that here, $u \in \mathcal{M}_0$ can be chosen to span the one-dimensional subspace $\mathcal{M}_0 \subset L^2(\mathbb{R}^n)$ (see Lemma 3.4), hence $u(x) = e^{-\frac{(x,x)}{2}},\ x \in \mathbb{R}^n$. Now to prove Theorem 1.1, let $x \in U \subset L$ and $s, \overline{s}$ as chosen above. Let $\gamma : [0,1] \to L$ be any closed smooth path in L with basepoint x, that is $\gamma(0) = \gamma(1) = x$ and let $\mathcal{P}_\gamma^Z(t) : (\hat{R}_L)_{\gamma(0)} \to (\hat{R}_L)_{\gamma(t)}$ be the parallel transport in $\hat{R}_L$ induced by Z, it follows that there is a unique smooth path

$$S : [0,1] \to U(n) = Sp(2n) \cap O(2n) \text{ s.t. } \mathcal{P}_\gamma^Z(t)(s(x)) = S(t).s(x),$$

so that $S(0) = Id$ where we used the trivialization of i^*R^J induced by the Euclidean connection ∇^0 on $T\mathbb{R}^{2n}$ to compare $\mathcal{P}_\gamma^Z(t)(s(x))$ and $s(x)$ for any $t \in [0,1]$ in i^*R^J. Analogously we have a path

$$\hat{S} : [0,1] \to \hat{U}(n) \text{ s.t. } \mathcal{P}_\gamma^{\overline{Z}}(t)(\overline{s}(x)) = \hat{S}(t).\overline{s}(x),$$

so that $\rho_2(\hat{S}(t)) = S(t),\ t \in [0,1]$ and $\hat{S}(0) = Id_{Mp(2n)}$ where again, we used the triviality of i^*P^J induced by the Euclidean connection ∇^0 on $T\mathbb{R}^{2n}$ to compare $\mathcal{P}_\gamma^{\overline{Z}}(t)(s(x))$ and $s(x)$ in i^*P^J. By the construction of Z in (36), it follows that $S(1) \in i(O(n))$, where $i : O(n) \hookrightarrow U(n)$ is the inclusion defined in (34). So writing $S(1) = \left(\begin{smallmatrix} A & 0 \\ 0 & A \end{smallmatrix}\right)$ for $A \in O(n)$ we have using (20) and setting $s(x) = p$

$$\begin{aligned} \mathcal{P}_\gamma^{\nabla^g}(1)[p,u] &= [\mathcal{P}_\gamma^{\overline{Z}}(1)p, u] = [p, \kappa(\hat{S}(1))u] = [p, (\kappa(g(A), m))u(y)] \\ &= [p, \det(A)^{\frac{1}{2}}u(A^t y)] = [p, i^m u(y)],\ y \in \mathbb{R}^n, \end{aligned} \tag{37}$$

where, as above, $m \in \mathbb{Z}_4$ is determined by requiring $\det(A)^{\frac{1}{2}} = |\det(A)|i^m$ and we used that $|\det(A)| = 1$. On the other hand, since $S(1) = \left(\begin{smallmatrix} 0 & A \\ -A & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ we have the decomposition

$$\kappa(\hat{S}(1)) = \hat{S}_{W,m'}\hat{S}_{W',n}, \quad \text{where } W = (0, A^t, 0),\ W' = (0, -Id_{R^n}, 0), \tag{38}$$

where $\hat{S}_{W,m}, \hat{S}_{W',n} \in \mathcal{U}(L^2(\mathbb{R}^n))$ are as referred to in Theorem 3.2 and $m' \in \mathbb{Z}_4$ is determined by using (25) and noting that $P = Q = P' = Q' = 0$, then

$$\begin{aligned}(\hat{S}_{W,m'}\hat{S}_{W',n}u)(x) &= \kappa(t(P))\kappa(g(A^t), m')\hat{J}\kappa(t(Q))\kappa(t(P'))\kappa(g(-Id_{\mathbb{R}^n}), n)\hat{J}\kappa(t(Q'))u(x) \\ &= i^{n-n}\kappa(g(A^t), m')u(x) = i^{m'}u(x),\ x \in \mathbb{R}^n,\end{aligned} \tag{39}$$

where we used $\hat{J}u = i^{-\frac{n}{2}}u$ (recall that $u(x) = e^{-\frac{(x,x)}{2}}$) and we fixed $\det(-Id_{\mathbb{R}^n})^{\frac{1}{2}} = i^n$, so by comparing with (37) we infer that $m = m' \in \mathbb{Z}_4$. Now, by using notation from Theorem 3.2, we have $\hat{\mu}(\hat{S}(1)) = \hat{\mu}(\hat{S}_{W,m'}) + \hat{\mu}(\hat{S}_{W',n}) = 2m - n + 2n - n = 2m$ so comparing that to (28) resp. (18) we deduce that

$$2m = \hat{\mu}(\hat{S}(1))) = \mu_{L_0,2}(\hat{S}(1)) = 2\mu_{CLM}([L(1)], L(t)) \bmod 8, \tag{40}$$

where $L(t) = S(t)L_0$ and $L_0 = \{0\} \times \mathbb{R}^n$ and we used that $\dim(L(0) \cap L(1)) = n$. By definition $L(t) = \mathrm{span}\{\sum_{i=1}^{2n} S_{ij}(t)e_i\}_{j=1}^n$, where $\mathrm{span}\{e_j\} = L_0$ denotes the standard basis of $\{0\} \times \mathbb{R}^n$, so defining $\tilde{S} \in Sp(2n)$ by $\tilde{S}e_j = s(x)_j$ if $\mathrm{span}_{j=1}^n s(x)_j = T_xL$ implies that $\tilde{S}L(t) = \mathrm{span}\{\sum_{i=1}^{2n} S_{ij}(t)s(x)_i\}_{j=1}^{2n}$ which means that $\tilde{S}L(t) = i_*(T_{\gamma(t)}L)$, if $i : L \hookrightarrow M$ is the inclusion. Using (40) and the invariance of μ_{CLM} under symplectic mappings we arrive at

$$m = \mu_{CLM}(\tilde{S}[L(1)], \tilde{S}L(t)) = \mu_{CLM}(i_*(T_xL), i_*(T_{\gamma(t)}L)),$$

which is by (37) exactly the content of Theorem 1.1. Now to prove Corollary 1.2, note that if $\mathrm{Hol}_p(\overline{Z})$ is the holonomy group of $p \in (\hat{P}_L)_x,\ x \in L$, that is

$$\mathrm{Hol}_p(\overline{Z}) = \{g \in \hat{O}(n) : \exists \gamma : [0,1] \to L,\ \gamma(0) = \gamma(1) = x,\ g.\gamma_p(0) = \gamma_p(1)\},$$

then there is an identification

$$\mathrm{Par}^{\nabla^g}(i^*Q) := \{\phi \in \Gamma(i^*Q_0) : \nabla\phi = 0\} \simeq \{u \in \mathcal{M}_0 : \kappa(\mathrm{Hol}_p(\overline{Z})u = u\}$$

for any $p \in \hat{P}_L$. Since we have shown above that $\kappa(\mathrm{Hol}_p(\overline{Z})) = \mathbb{Z}_4 \subset S^1$ if $S^1 \subset \mathbb{C}$ acts by multiplication on $\mathcal{M}_0$ and $\mathcal{M}_0$ has complex dimension one, we infer by the homotopy invariance of μ_{CLM} that $\mathrm{Par}^{\nabla^g}(i^*Q) = 1$ if and only if $\mu_{CLM}(i_*(T_xL), i_*(T_{\gamma(t)}L)) = 0 \mod 4$ for any $\gamma \in \pi_1(L)$, which proves the Corollary.

To prove Theorem 1.3, choose $p \in (\hat{P}_L)_x,\ x \in L$ and extend $\delta_p \in i^*(\mathcal{Q}^J)'_x = i^*\mathcal{Q}'_x$ using the connection ∇^0 on $i^*\mathcal{Q}'$ induced by the canonical flat connection of $i^*(T\mathbb{R}^{2n})$ to an element $\delta_p \in \Gamma(i^*\mathcal{Q}')$. Now write $\delta_p(x) = [p, \delta(0)]$ and consider the parallel transport $\mathcal{P}_\gamma^{\nabla^g}$ along $\gamma : [0,1] \to L,\ \gamma(0) = x, \gamma(1) = y$ for $y \in L$ induced by the Levi-Civita-connection ∇^g of L in $i^*\mathcal{Q}'$. Then

$$\mathcal{P}_\gamma^{\nabla^g}(1)[p, \delta(0)] = [\mathcal{P}_\gamma^{\overline{Z}}(1)p, \delta(0)] = [p, \kappa(\hat{S}(1))\delta(0)] \tag{41}$$

where as above, $\hat{S} : [0,1] \to \hat{U}(n)$ is determined by the requirement that $\mathcal{P}_\gamma^{\overline{Z}}(t)(p) = \hat{S}(t).p$, where again we used the trivialization of i^*P^J induced by ∇^0 to consider p as an element of $(i^*\mathcal{Q}')_{\gamma(t)}$ for any $t \in [0,1]$ and $\hat{S}$ lifts the path $S : [0,1] \to U(n)$ determined by $\mathcal{P}_\gamma^Z(t)(r) = S(t).r$, where $r = \hat{f}(p) \in (\hat{R}_L)_x$. Assume now that $i_*(T_xL) \cap i_*(T_yL) = \emptyset$ in $\mathbb{R}^{2n}$, then since

$$S(1) = \left(\begin{smallmatrix} A & -B \\ B & A \end{smallmatrix}\right), \det(B) \neq 0,$$

where $A, B \in M(n, \mathbb{R}),\ A^tA + B^tB = I$ and A^tB symmetric, we can write following Theorem 3.2 resp. Gosson [8] (Chapter 7.1)

$$\kappa(\hat{S}(1)) = \hat{S}_{W,m} \quad \text{where } W = (P, L, Q) = (-AB^{-1}, -B^{-1}, -B^{-1}A),$$

and explicitly

$$\hat{S}_{W,m} = \kappa(t(-AB^{-1}))\kappa(g(-B^{-1}), m)\hat{J}\kappa(t(-B^{-1}A)). \tag{42}$$

So we get by applying (32) and (23) and by setting $u_\epsilon = \frac{1}{\epsilon^n} u(\frac{x}{\epsilon})$ with u as in (39) and $f \in \mathcal{S}(\mathbb{R}^n)$

$$\begin{aligned}(\hat{S}_{W',m'}\delta(0))(f) =&\lim_{\epsilon\to 0}\int_{\mathbb{R}^n} \overline{\hat{S}_{W',m'}u_\epsilon} f\,dx \\ =&(\frac{1}{2\pi})^{n/2} i^{-m'+n/2}|\det(-B^{-1})|^{\frac{1}{2}}.\end{aligned} \tag{43}$$

Now note that the definition $\hat{\mu}(\hat{S}_{W,m}) = 2m-n$ and the formula (27) for $\hat{\mu} : Mp(n) \to \mathbb{Z}_4$ are compatible, that is if $\hat{S} \in Mp(2n)$ and $\rho_2(\hat{S}) = S$ with $S = \left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right),\ \det(B) \neq 0$, then $\kappa(\hat{S}) = \hat{S}_{W,m}$ for some $W = (P, L, Q),\ m \in \mathbb{Z}_4$ and $\hat{\mu}(\hat{S}) = \hat{\mu}(\hat{S}_{W,m}) = 2m-n$. This follows from Theorem 7.22 (i) in Gosson's book ([8]). So, combining (43) with (41), setting $c(y) := (\frac{1}{2\pi})^{n/2}|\det(-B^{-1})|^{\frac{1}{2}}$ and using (18) together with $2m'-n = \hat{\mu}(\hat{S}_{W',m'})$, we arrive at formula (4) in Theorem 1.3. To examine the case $i_*(T_xL) = i_*(T_yL)$, note that in this case $\kappa(\hat{S}(1))$ in (41) decomposes as in (38) and consequently

$$\begin{aligned}\kappa(\hat{S}(1))\delta(0) &= (\hat{S}_{W,m'}\hat{S}_{W',n})\delta(0) \\ &= \kappa(g(A^t), m')\hat{J}\kappa(g(-Id_{\mathbb{R}^n}), n)\hat{J}\delta(0) \\ &= i^{n-n}\kappa(g(A^t), m')\delta(0) = i^{-m'}\delta(0).\end{aligned} \tag{44}$$

Since $\hat{\mu}(\hat{S}(1)) = \hat{\mu}(\hat{S}_{W,m'}) + \hat{\mu}(\hat{S}_{W',n}) = 2m'-n+2n-n = 2m'$ we can use again (18) to arrive at (5). Now to proof (6) in Corollary 1.4, note that if $\hat{S}(1) \in Mp(2n)$ is as in (41) and that $\mathcal{M}_0 \subset L^2(\mathbb{R}^n)$ is spanned by $u(x) = e^{-\frac{(x,x)}{2}},\ x \in \mathbb{R}^n$, then with $m' \in \mathbb{Z}_4$ as in (43) we have

$$\begin{aligned}\delta(0)(\kappa(\hat{S}(1))u)(x) = \overline{(\kappa(\hat{S}(1))^*\delta(0))(u)}(x) &= (\tilde{c}(y)i^{m'-n/2}\mathbf{1})(u)(x) \\ &= \tilde{c}(y)i^{m'-n/2}\int_{\mathbb{R}^n} u(x)dx = \tilde{c}(y)i^{m'-n/2}2\pi^{n/2}\end{aligned}$$

where $\tilde{c}(y) \in \mathbb{R}^+$ and we used that $\hat{S}^*_{W,m'} = \hat{S}_{\tilde{W},n-m'}$, for some quadratic form $\tilde{W}$ (cf. [8], Prop. 7.2). Since $\hat{S}(1) \in \hat{U}(n)$, we have $\kappa(\hat{S}(1))u = \hat{c}u$ for some $\hat{c} \in U(1)$ so we see that $\tilde{c}(y)2\pi^{n/2} = 1$ and $\hat{c} = i^{m'-n/2}$, which gives (6) by the arguments given below (43). Finally (7) follows by noting that $\kappa(\hat{U}(n))(\mathcal{M}_l) \subset \mathcal{M}_l,\ l \in \mathbb{N}_0$ (see 3.4), using

$$\delta(0)(\kappa(\hat{S}(1))u) = \overline{\kappa(\hat{S}(1))^*\delta(0))\overline{u}} = i^{m'}u(0),$$

where $u \in \mathcal{M}_l$, $\hat{S}(1),\ m' \in \mathbb{Z}_4$ are as in (44) and finally using (18) again.

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